The Stability of a Vertical Flow Which Arises From Combined Buoyancy Modes

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We consider a mass transfer induced buoyancy force, superimposed on a thermally induced one, in a flow generated adjacent to a vertical surface. Flow and stability were investigated for a Prandtl number of 0.7; for Schmidt numbers of 2.0, 0.94, and 0.2; and for different magnitudes of the buoyancy effects.

SCOPE

Mass transfer occurring in a body of fluid gives rise to a buoyancy force if the concentration gradient causes density differences. If the concentration of the diffusing species is sufficiently small, the equations governing the phenomenon are identical to those governing a thermally induced flow. A frequently occurring circumstance in our environment and in technological applications is the simultaneous transport of both thermal energy and a chemical constituent. The two buoyancy effects may be in the same direction or opposing. They may even have different spatial extent in the fluid if the Prandtl and Schmidt numbers are different.

The consequences, the effects on the resulting flow and on its stability to disturbances whose growth determines transition to turbulence, were investigated for a gas. Mass transfer effects were determined over the known range of mass transfer rates, that is, of Schmidt number.

The analyses predict transport rates as well as the conditions of instability and the actual disturbance growth rates. These results amount to an assessment of those complicated and interacting effects over a wide range of practical applications. Previous developments are summarized in a later section, preceding the analysis.

CONCLUSIONS AND SIGNIFICANCE

The occurrence of a second buoyancy inducing transport process may cause major alterations of the stability characteristics of the resulting flow. There are very complex interactions between disturbances in velocity, temperature, and concentration. For a range of Lewis number, and of the ratio of the two buoyancy components, the additional buoyancy mode alters the disturbance growth characteristics and shifts the frequency range of the disturbances which are most highly amplified in actual flows.

This frequency shift is most apparent when the stability characteristics are expressed purely in terms of the thermal Grashof number. This corresponds to a flow circumstance in which the temperature difference ($t_0 - t_\infty$) is held constant while one observes the influence of an added mass transfer on disturbance filtering and growth. However, if we discuss the amplification in terms which more accurately reflect the vigor of the flow, for example, with a Grashof number composed of the unweighted sum of the thermal and chemical buoyancy components, the shift progressively disappears as the Lewis number approaches 1. However, the effect remains for $Le = 2.9$ and 0.29, and, in addition, the flow is less stable when the two buoyancy effects are opposed.

Our results are expected to be realistic estimates of the stability characteristics of actual flows because of the detailed past success of linear stability theory, compared to many experiments, in its predictions of disturbance growth rates, filtered frequency, etc., in flows, both in an unstratified ambient medium and, most recently, by Jaluria and Gebhart (1974*b*) in a stratified one. The small downstream variation found here in disturbance form lead us to expect that the calculations of disturbance amplitude are again realistic.

There is the remaining question of how we should accurately estimate the effective local vigor of a flow induced by combined buoyancy modes. The effect of Lewis number on the respective spatial extents of the boundary layers, and the differing resultant modification of the form of the velocity field, makes this a very complicated and still unanswered question.

The methods used here may be directly applied to other important combined buoyancy mode flows. Both bounded and unbounded flows of great practical importance arise through combined modes and in many different fluids and applications. These questions may also be further pursued to a better understanding of the subsequent events which lead to the important matter of transition to the much more effective turbulent regime.

PREVIOUS WORK

One of the first investigations of the flow resulting from combined buoyancy modes was by Somers (1956), who applied integral method analysis to predict the flow that would arise adjacent to a wetted isothermal surface. Mathers et al. (1957) and Wilcox (1961) using, respectively, an analogue and an integral analysis, follow Somers in assigning a factor $\sqrt{Sc}$ to the mass transport contribution to the total buoyancy term in the momentum equation. Such

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calculations gave transport coefficients within 20% of experimentally measured ones. Later investigations have shown that integral methods are not sufficiently precise when Pr and Sc are very different, that is, when the Lewis number $Le = Sc/Pr$ is much different from 1.

By using the boundary layer equations and by including both buoyancy effects, similar solutions may also be found. Gebhart and Pera (1971, 1972) determined the variables, parameters, and equations governing a class of surface conditions and calculated the pattern of several flows resulting from simultaneous thermal and mass transfer. They also showed that the normal velocity component at the surface is negligible over a wide range of practical conditions found both in the atmosphere and in bodies of water. More recently, Bottemanne (1972) reported measurements made on a vertical cylinder, simultaneously transferring both heat and water vapor into air. The agreement with a numerical solution, Bottemanne (1971), in terms similar to those of Gebhart and Pera (1971), is said to be within experimental accuracy, reportedly 6%.

The earliest application of stability theory to natural convection flows was by Plapp (1957), whose asymptotic expansion, valid only for large values of the Grashof number, led to estimates of neutral stability at great variance with observed appearances of instability. Kurtz and Crandall (1962) numerically calculated the neutral curve from the uncoupled disturbance vorticity equation, that is, neglecting disturbance coupling through buoyancy. Nachtsheim (1963) included coupling, for a uniform temperature vertical surface, for the Prandtl number of both air and water. These results were in qualitative agreement with the hot-wire data of Colak-Antic (1964) in air. Szewczyk's dye trace results in water (1962), and much other similar data are now known to be compatible with extensive coupled results of Nachtsheim and of Hieber and Gebhart (1971*a*).

A series of analytical and experimental studies by Polymeropoulos and Gebhart (1967), Knowles and Gebhart (1968, 1969), Dring and Gebhart (1968, 1969), Hieber and Gebhart (1971*a*, 1971*b*), Jaluria and Gebhart (1973, 1974*a*, 1974*b*), summarized in part by Gebhart (1971, 1973*a*, 1973*b*), further clarified disturbance growth mechanisms and transition to turbulence of flows along vertical surfaces. All of these results have confirmed in detail the appropriateness of linear stability theory in the study of the early growth mechanisms. The results already available from stability theory have been extended by Gebhart and Pera (1971) to flows resulting from combined thermal and mass transfer for equal Prandtl and Schmidt numbers.

Here we have investigated such effects for several values of the Lewis number, for buoyancy both aiding and opposing, and for flows adjacent to a vertical isothermal and isoconcentration surface. Stability characteristics are found to be strongly affected by the resultant changes in the velocity and temperature distributions and by the third disturbance component, that due to mass transfer. It appears as a second coupling term in the disturbance vorticity equation.

ANALYSIS OF THE STEADY LAMINAR FLOW

The appropriate equations result from the conservation of force and momentum, mass energy, and the diffusing species. The effect of viscous dissipation and pressure are neglected in the energy balance, as they commonly are for the terrestrial magnitude of the gravity force. The molecular transport properties μ, k, and D are taken as constants. The concentration of C of the diffusing species is assumed small compared to that of the other species present in the surrounding field. This is notably the case for certain very important circumstances: water vapor or carbon dioxide diffusing in atmospheric air or salts and contaminants diffusing in terrestrial water. This hypothesis allows us to assume Fick's law of diffusion, to ignore the effects of diffusion in the energy equation, and to retain only the linear term of the concentration buoyancy effect in the momentum equation, the analogue of the Boussinesq approximation, also invoked here. We may also ignore the normal component of velocity at the surface which is generating the flow when a flow adjacent to a surface is considered.

For plane flow, the vertical coordinate x is taken positive from the initial of flow, and y is normal thereto. With the usual boundary layer approximations and the limitations invoked above, the equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta(t - t_\infty) + g\beta^*(C - C_\infty)$$

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \alpha_t\frac{\partial^2 t}{\partial y^2}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2}$$

where ν, α_t, and D are the molecular diffusion coefficients. For an ideal gas, $\beta^* = \frac{1}{p}\left(\frac{M_1}{M_2} - 1\right)$, where M_2 is the molecular weight of the diffusing species and M_1 is characteristic of the remaining gaseous species.

Physical quantities are generalized by the use of a stream function $\Psi(x, y)$ and by the differences $(t_o - t_\infty)$ and $(C_o - C_\infty)$ locally between the midplane $(y = 0)$, the surface, and the distant fluid. These functions are expressed in terms of variable $\eta(x, y)$ as

$$\eta = y\ b(x)$$

$$\Psi(x, y) = \nu c(x)\ f(\eta)$$

$$T(\eta) = \frac{t - t_\infty}{t_o - t_\infty}; \quad C(\eta) = \frac{C - C_\infty}{C_o - C_\infty}$$

It may be shown that $b(x)$ and $c(x)$ yield ordinary differential equations in $f(\eta)$, $T(\eta)$, and $C(\eta)$ when the quantities $(t_o - t_\infty)$ and $(C_o - C_\infty)$ have a power law dependence on x. Similarity will require that these two quantities have the same power law variation, that is

$$t_o - t_\infty = N_t\, x^n; \quad C_o - C_\infty = N_c\, x^n$$

where $-0.6 \leq n \leq 1.0$ for physical realism, see Gebhart (1973*a*).

We take the values t_∞ and C_∞ as constants, that is, no thermal or species concentration stratification, although similarity does result for power law conditions of stratification. Thus, $n = 0$ corresponds to the isothermal and isoconcentration surface. A general discussion of these considerations is given by Gebhart (1973*a*, 1971).

With the same power law x dependence of $(t_o - t_\infty)$ and $(C_o - C_\infty)$, similarity was formulated by Gebhart and Pera (1971) as

$$b(x) = \frac{G}{4x}; \quad c(x) = G;$$

where:

$$G = 4\left(\frac{PGr_{x,t} + QGr_{x,c}}{4}\right)^{1/4} = 4\left(\frac{Gr'}{4}\right)^{1/4}$$

$$Gr_{x,t} = \frac{g\beta x^3\ (t_o - t_\infty)}{\nu^2}$$

$$Gr_{x,c} = \frac{g\beta x^3 (C_o - C_\infty)}{\nu^2}$$

or

$$G = 4\left(\frac{Gr_{x,t}}{4}\right)^{1/4} (P + QN)^{1/4}$$

$$N = \frac{Gr_{x,c}}{Gr_{x,t}} = \frac{\beta^* N_c}{\beta N_t}$$

P and Q are any constants such that Gr' above is positive. The parameter N is a measure of the relative importance of the two buoyancy force components. The two transfer processes are also characterized by the Prandtl and Schmidt numbers. An estimate of the relative spatial extent of the thermal and concentration layers, around $Le = 1.0$, is

$$\frac{\delta_t}{\delta_c} = 0\left(\sqrt{\frac{Sc}{Pr}}\right) = 0\left(\frac{1}{\sqrt{Le}}\right)$$

The distribution of the net buoyancy force across the boundary region is governed by N, Pr, and Sc through the coupling which appears in Equations (1), (2), and (3) below. As a result of these interactions, we do not know, in a circumstance where the two effects are opposed ($N < 0$) and for $Le \neq 1.0$, the conditions that ensure that the total buoyancy force remain positive over the whole boundary region. We also do not know a priori if the calculated flow will be locally reversed or otherwise physically unreasonable, that is, in terms of the approximations made. Clearly, N may assume any positive or negative value since β, β^*, $(t_o - t_\infty)$, and $(C_o - C_\infty)$ may each be positive or negative.

For any choice of P and Q, in which they both are not zero, we see that G is no longer defined when N has the value

$$N = N^* = -P/Q$$

One might change P and Q in order to study a large range of N subject to certain considerations of realism in such applications of the boundary layer approximations. However, for convenience, the values used in the present work are

$$P = 1; \quad Q = 0$$

Thus, G is a function only of the thermal Grashof number. However, no information is lost, as $Gr_{x,t}$ and $Gr_{x,c}$ are related by N which will appear in the differential equations. The mass transfer process appears there as N and Sc dependent modifications of the heat transfer process.

With $P = 1$ and $Q = 0$, $Gr_{x,t}$ must be positive. This is ensured by taking the x axis positive in the direction of the gravity if $\beta(t_o - t_\infty)$ is negative, and in the opposite direction if it is positive. This is appropriate, since the largest negative value of N we consider, -0.5, does not give rise to any flow reversal for the Prandtl and Schmidt number range used. Incidentally, if there is simultaneous transfer of several dilute chemical species i, multiple parameters

$$N_i = \frac{\beta_i^* N_{ci}}{\beta N_t}$$

appear in the differential equations, but the procedure is unchanged.

In terms of the variables so defined, and for an isoproperties surface condition ($n = 0$), the differential equations and boundary conditions of interest here are written below. Similarity is then seen to result; that is, Φ, C and f depend on x and y only through $\eta(x, y)$:

$$f''' + 3ff'' - 2f'^2 + T + NC = 0 \quad (1)$$

$$T'' + 3Pr\, fT' = 0 \quad (2)$$

$$C'' + 3Sc\, fC' = 0 \quad (3)$$

$$f(0) = f'(0) = 1 - T(0) = 1 - C(0) = T(\infty) = C(\infty) = f'(\infty) = 0 \quad (4)$$

The condition $f(0) = 0$ is obtained by letting the normal velocity component at $y = 0$, $v(x, 0)$, associated with the mass transfer process, be negligible. This approximation is valid when

$$\frac{v(x, 0)x}{\nu} << G$$

or, equivalently, when

$$\frac{1}{Sc}\,\frac{(C_o - C_\infty)}{\rho}\,[-C'(0)] << 1$$

Otherwise, $v(x, 0)$ must vary as $x^{-1/4}$ in order to result in a boundary condition $f(0) =$ constant $\neq 0$ which would preserve similarity.

Equation (2) indicates that if f goes to a negative constant as η goes to infinity, T diverges. Thus, $f(\infty) = f_\infty$ must be positive. Its value is related to local mass flow rate in the boundary region, at x, by

$$\dot{m} = \int_0^\infty \rho\, u\, dy = \nu\rho\, G f_\infty$$

If, however, f_∞ is found in the numerical integration to tend to a negative value, it is because the chemical buoyancy is opposing the thermal one and is dominant. Thus, the chosen positive x axis direction is improper, since there must be a positive mass flow rate for the boundary layer formulation.

Integrating (2) and (3) from $\eta = 0$ and $\eta = \infty$, we find

$$\left[\frac{-T'(0)}{3f_\infty Pr}\right]^{1/Pr} = \left[\frac{-C'(0)}{3f_\infty Sc}\right]^{1/Sc}$$

where f_∞ in this relation must be positive; that is, the positive x direction must be chosen in the direction of net flow.

Transport parameters and the tangential velocity component are found to be

$$Nu_x = -\frac{T'(0)}{\sqrt{2}}\,\sqrt[4]{Gr_{x,t}}$$

$$Sh_x = -\frac{C'(0)}{\sqrt{2}}\,\sqrt[4]{Gr_{x,t}}$$

$$u(x, y) = \frac{2}{x}\,\nu f'(\eta)\,\sqrt[2]{Gr_{x,t}}$$

Calculations by a fourth-order Runge-Kutta method were performed for $Pr = 0.7$, for each $Sc = 0.94$, 2.0, and 0.2. These values correspond to the transfer, into air, of carbon dioxide, ethyl benzene, and hydrogen, respectively. These span the known Schmidt number range. Positive and negative values of N were considered. However, our primary interest here is the stability of such flows, and one may refer to Gebhart and Pera (1971) for more extensive base flow considerations. Some of our velocity, temperature, and concentration profiles are given in Figures 1, 2, and 3, and the parameters of interest are listed in the Appendix. Note that the curves for $N = 0$ indicate the distributions resulting from a purely heat transfer induced buoyancy. The effect of Sc and N are seen by comparison. The step size and the range of integration were such that these results are expected to be within 1% of the exact values.

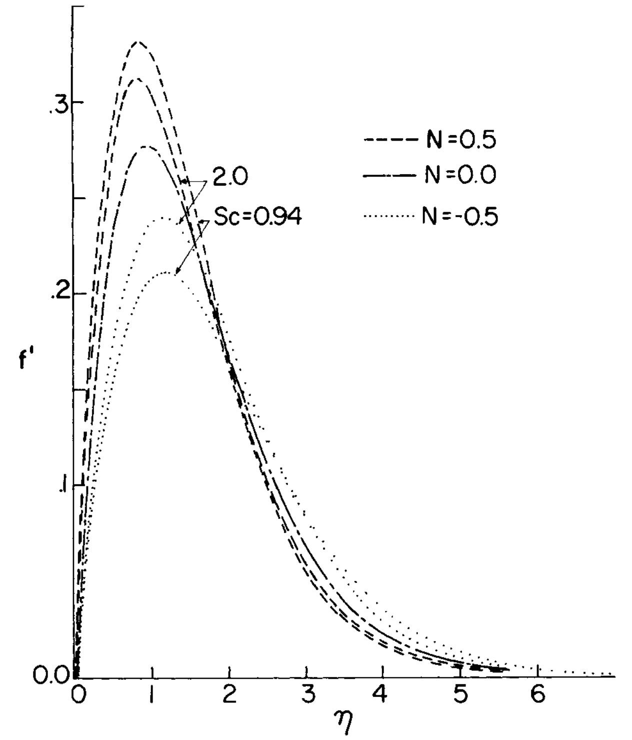

Fig. 1. Velocity distributions for $Pr = 0.7$ and for $Sc = 0.94$ and 2.0, compared with that for $N = 0$.

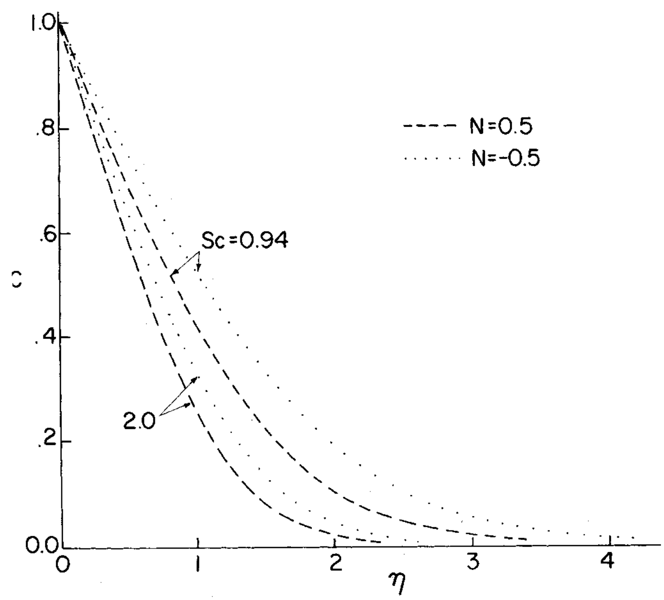

Fig. 3. Concentration distributions for $Pr = 0.7$ and for $Sc = 0.94$ and 2.0.

STABILITY ANALYSIS

The stability of the flow is examined by superimposing a similar small disturbance in stream function, temperature, and concentration on the flow and then by determining whether they locally amplify or damp. The assumed disturbance forms are

$$\Psi'(x, y, \tau) = \nu G\phi(\eta) \exp [(\alpha_d x - \omega_d \tau)]$$

$$t'(x, y, \tau) = (t_o - t_\infty)\ s(\eta) \exp [i(\alpha_d x - \omega_d \tau)]$$

$$C'(x, y, \tau) = (C_o - C_\infty)\ d(\eta) \exp [i(\alpha_d x - \omega_d \tau)]$$

where ϕ, s, and d are the complex amplitude functions. We take α_d as complex and ω_d as real, so that ω_d represents disturbance frequency, $(\alpha_d)_r = 2\pi/\lambda$ is the wave number, and $-(\alpha_d)_i$ is the exponential spatial amplification rate, where r means real and i imaginary.

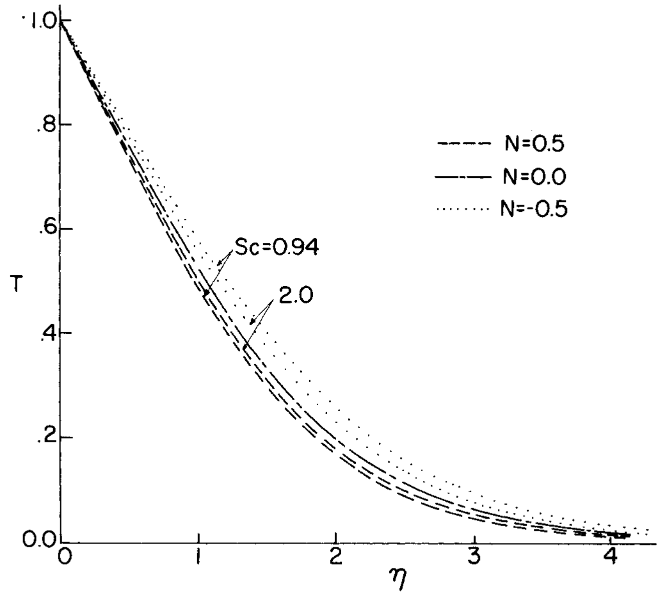

Fig. 2. Temperature distributions for $Pr = 0.7$ and for $Sc = 0.94$ and 2.0, compared with that for $N = 0$.

The resulting stability equations, after the usual procedure of linearization and nondimensionalization, are, in the paralleled flow approximation

$$i\alpha G(f' - c)(\phi'' - \alpha^2\phi) - f'''\ \phi = \phi'''' - 2\alpha^2\phi'' + \alpha^4\phi + s' + Nd'$$

$$i\alpha GPr(f' - c)s - T'\phi = s'' - \alpha^2 s \qquad (5)$$

$$i\alpha GSc(f' - c)d - C'\phi = d'' - \alpha^2 d$$

where the primes indicate derivatives with respect to η, and where

$$\alpha = \alpha_d\delta \quad \omega = \frac{\omega_d\delta}{U} \quad c = \frac{\omega}{\alpha}$$

$$U = \frac{\nu G^2}{4x} \quad \delta = \frac{4x}{G} \qquad (6)$$

Equation (5) is the usual Orr-Somerfeld equation, with the two coupling terms s' and d' which arise from the buoyancy effects of the temperature and concentration disturbances. The system is seen to be of eighth order.

The disturbances are assumed to vanish at a large distance out from the surface generating the flow, and we have the four conditions

$$\phi(\infty) = \phi'(\infty) = s(\infty) = d(\infty) = 0 \qquad (7)$$

Since the disturbance amplitudes are zero at a surface of large relative capacity, we also have

$$\phi(0) = \phi'(0) = s(0) = d(0) = 0 \qquad (8)$$

These last two conditions are particular forms of more general ones

$$s(0) = K_1 s'(0), \qquad d(0) = K_2 d'(0)$$

where K_1 and K_2 depend on the relative thermal and chemical species capacities at the surface, compared with the adjacent fluid. Knowles and Gebhart (1968) investigated this condition in detail for thermal disturbances.

The disturbance equations have been numerically integrated in their complete form, that is, with the coupling terms, by using the boundary conditions (7) and (8). The method used was developed by Hieber and Gebhart (1971*a*). The solutions are taken as the sum of four linearly independent integrals which decay exponentially at large η, each perhaps at a different rate. Each integral is

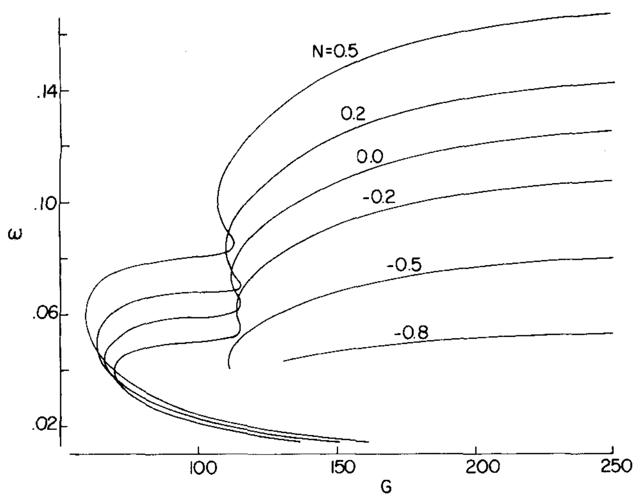

Fig. 4. Neutral curves for $Pr = 0.7$ and $Sc = 0.94$ (carbon dioxide in air) in terms of the thermal Grashof number.

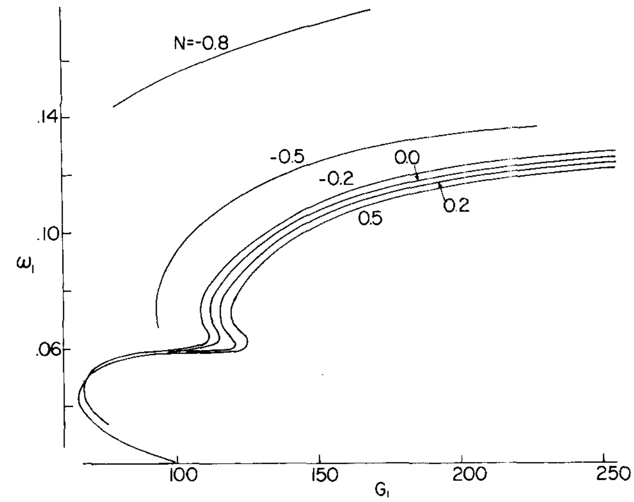

Fig. 5. Neutral curves for $Pr = 0.7$ and $Sc = 0.94$ (carbon dioxide in air) in terms of the thermal Grashof number.

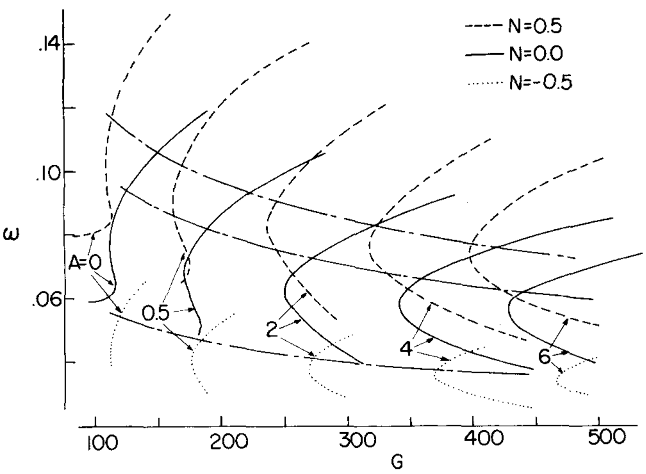

Fig. 6. Downstream disturbance amplification for $Sc = 0.94$, in terms of G.

integrated inward from the appropriate asymptotic behavior and thus satisfies the boundary conditions at infinity. The sums are

$$\phi = \phi_1 + B_2\phi_2 + B_3\phi_3 + B_4\phi_4 \quad (9a)$$

$$s = s_1 + B_2 s_2 + B_3 s_3 + B_4 s_4 \quad (9b)$$

$$d = d_1 + B_2 d_2 + B_3 d_3 + B_4 d_4$$

where B_2, B_3, B_4 were chosen so that three of the four boundary conditions at $\eta = 0$ are satisfied. The fourth one is satisfied only if α takes on an appropriate value (eigenvalue) for the given G and ω. Typically, α is determined by iteration, for example, by linear interpolation of the results of two calculations with slightly different values of α.

In a fourth-order Runge Kutta method, the step size was halved until the resulting change in α was insignificant. Similarly, the value of η_{edge}, where the asymptotic behavior at infinity was used, was increased until an insignificant change in α was obtained. The values calculated are thought to approximate the exact ones within 1%.

RESULTS

The results were plotted on two kinds of stability planes. The first is in terms of G, that is, with $P = 1$ and $Q = 0$, where the frequency ω_d has also been made nondimensional with G and is called ω. The second presentation uses a total Grashof number

$$G_1 = 4\sqrt[4]{\frac{Gr_{x,t} + Gr_{x,c}}{4}} = G(1+N)^{1/4} \quad (10)$$

which corresponds to $P = 1$ and $Q = 1$. The frequency, generalized with G_1, is called ω_1, and becomes

$$\omega_1 = \omega(1+N)^{-3/4} \quad (11)$$

The base flow analysis has been discussed in terms of G, and it may be the most useful parameter in an experimental verification, since temperature is more easily measured than concentration. However, G does not in general give a proper measure of the vigor of the flow. Although G_1, with the unweighted sum of the two buoyancy effects, is more appropriate, this is still not the proper choice for $Pr \neq Sc$. The proper weighting by $P \neq Q$ is known only from estimates through integral analysis. These are known to be unreliable for large and small values of the Lewis number, $Le = Sc/Pr$.

Neutral curves for $Pr = 0.7$, $Sc = 0.94$, and six values of N, from -0.8 to $+0.5$, are seen in Figure 4, in terms of G. Increasing mass transfer buoyancy upward, that is, $N > 0$, strongly destabilizes the flow in terms of G. Although we would expect this, it is not conclusive, since G is not a uniformly good measure of the actual buoyancy force. However, each of these neutral curves strongly suggests the sharp frequency filtering first found in a purely thermally driven flow. Filtering amounts to selectively amplifying only certain components of a more complicated naturally occurring disturbance.

The coordinates of Figure 5, in terms of G_1, are much more appropriate and also quite reasonable, since $Le = 1.34$. The small effect on stability in the N range from -0.2 to $+0.2$ agrees with the conclusions of Gebhart and Pera (1971). A single neutral curve would result for $Pr = Sc$, for all N, when G_1 is used. Our small difference in Pr and Sc is first strongly felt for $N = -0.5$ and very strongly at $N = -0.8$. The effect for $N = -0.8$ is seen here as formally due to the singularity of the transformation of ω into ω_1 at $N = 1.0$. This singularity does not actually occur for $Le \neq 1.0$. That is, we should not take $P = Q$ for $Le \neq 1.0$ because the two transport processes have different spatial extents, and their simple sum does not properly represent the actual buoyancy effect, or whatever else is appropriate as their combined effect. We recall that for $Le = 1.0$ there is no flow for $N = 1.0$, no matter what values are assigned to P and Q.

The downstream $[G(x)]$ amplification rates are given by $-\alpha_i$. The ratio of the amplitude of any particular sinusoidal disturbance component downstream, at G, to that it had on crossing the neutral curve, at G_n, is given by Dring and Gebhart (1968) as

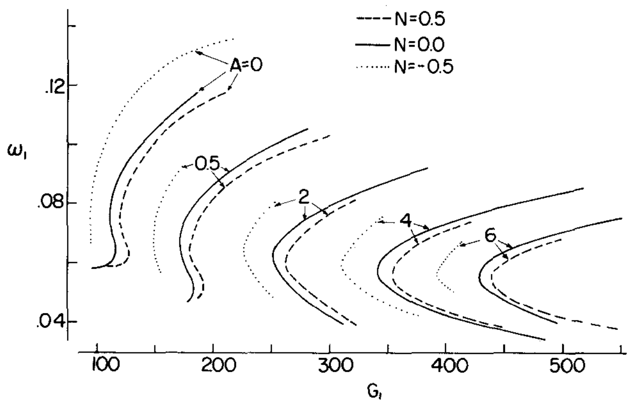

Fig. 7. Downstream disturbance amplification for Sc = 0.94, in terms of G_1.

$$R_{\rm amp} = \exp\left(-\frac{1}{3}\int_{G_n}^{G} \alpha_i\, dG\right) = \exp(A) \quad (12)$$

This is approximate to the extent that the form of the disturbance amplitude distributions across the boundary region changes downstream with G and is also subject to all other approximations already made. The above integration is performed in the ω, G plane along paths of constant physical frequency ω_d. This path is $\omega G^{1/3}$ = constant $\propto \omega_d$, or of $\omega_1 G_1^{1/3} \propto \omega_d$.

We have calculated $A(\omega, G)$ downstream across the band of frequency which experiences most rapid amplification, for both the Sc and N effects, for $Pr = 0.7$. Since the frequencies most rapidly amplified are not the earliest in G to be unstable, the notion of a critical G is again not important.

The results for $Sc = 0.94$ and $N = 0.5$, 0 and -0.5 are shown in Figure 6 in terms of G. Several constant physical frequency paths are shown, and the contours of A again show the sharp downstream frequency filtering first found by Dring and Gebhart (1968) for a purely thermally driven flow and since abundantly corroborated in experiment, see Gebhart (1973*b*). With the combined buoyancy modes, we see that disturbances are amplified more rapidly for increasing N, in terms of G. $N = +0.5$ and $N = -0.5$ appear to cause opposite effects of comparable amount in ω. Decreasing N appears to stabilize the flow and also to reduce the most highly filtered frequencies.

In the more reasonable ω_1, G_1 plane, Figure 7, the effects of N are greatly diminished, and decreasing N destabilizes. The location of the filtered band seems largely independent of N. However, we must interpret these results with care, since $P = Q = 1$ is still somewhat arbitrary, even for $Pr = 0.7$ and $Sc = 0.94$. Also, the proportionality factor between $\omega_1 G_1^{1/3}$ and ω_d depends on the particular values of N_t and N_c, not only on N.

However, these results for $Sc = 0.94$ do not amount to a demanding test of the effects of combined buoyancy modes on stability and disturbance growth mechanisms. The heat and mass transfer effects are approximately included by the parameter $(1 + N)$, as shown by Gebhart and Pera (1971) and confirmed above. Recall that for $Le = 1.35$ the boundary layers are of comparable extent.

For $Sc = 2$ ($Le = 2.9$,) the concentration boundary layer is relatively thin. Stability results for $Sc = 2.0$ and $N = -0.5$, 0 and $+0.5$ are seen in Figures 8 and 9 in terms of G and G_1, respectively. The effects of N on stability are much greater. In both representations, an opposing buoyancy effect destabilizes the flow and an aiding one stabilizes. These effects are large, and both are increased in going from the G to the more realistic G_1 representation. The destabilization for negative N is consistent with that found for $Sc = 0.94$. We note that for values of $Sc >$

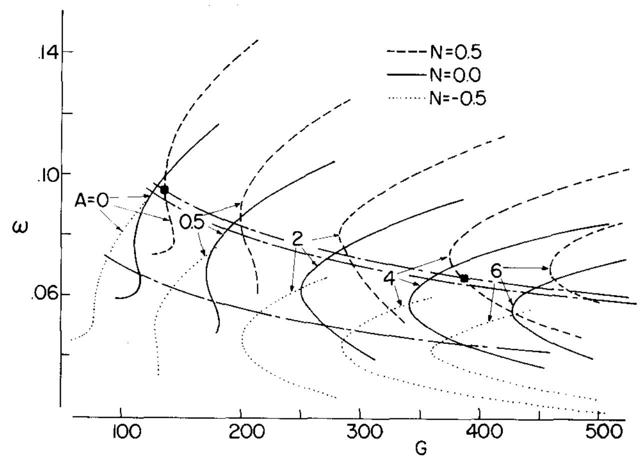

Fig. 8. Downstream disturbance amplification for Sc = 2.0, in terms of G.

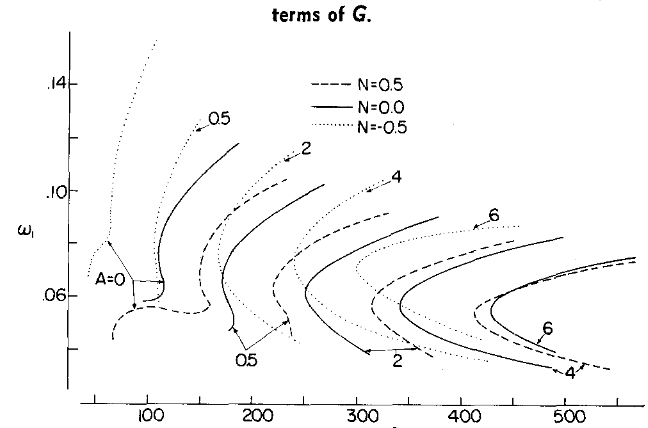

Fig. 9. Downstream disturbance amplification for Sc = 2.0, in terms of G_1.

Pr the concentration boundary layer is thinner than the thermal one.

We have also obtained amplification results for $Sc = 0.2$ ($Le = 0.29$) and $N = +0.2$. The results are limited to $A = 2$ because above this value the convergence of the numerical method became impractically slow for the ω range of the most amplified frequencies. Nevertheless, these results, in Figures 10 and 11, indicate that a positive N again stabilizes in both representations. No results were obtained for $N < 0$.

These stability results are initially surprising. Increasing positive N for any value of Sc and at any G means an increasingly vigorous and presumably more unstable flow. The results for $Sc = 0.94$ and 2.0 show stabilization instead. However, we recall that for $Le \neq 1.0$ the velocity is not simply increased over the whole range of η, but the distribution is also distorted. The major effect is at small η for $Le > 1.0$. This effect was found to be especially strong in the distribution of the viscous force f''', a very important term in the equations. It is very difficult to assess the effect of the added coupling term d'. Since its distribution is affected by coupling and by the Lewis number, it is difficult to conjecture the effect of a positive N when Le is not close to 1.

For typical values of N, G, and Sc, we calculated the distributions of the disturbance functions as the magnitude of the complex amplitude functions ϕ', s, d, that is, the x component of the velocity, the temperature, and the concentration disturbances, respectively. Each was normalized by its maximum value. The distribution of the phase of the x component of the velocity disturbance was also determined, since it might be useful in interpreting experiments.

The resulting figures are not shown here, for brevity, but they appear in Boura (1974). We will discuss the results

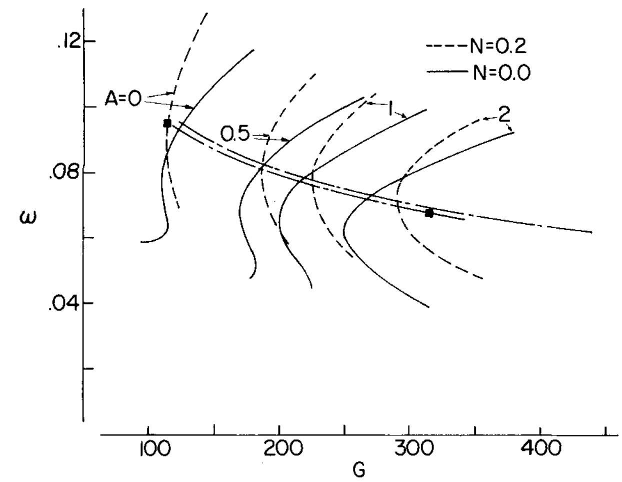

Fig. 10. Downstream disturbance amplification for $Sc = 0.2$, in terms of G.

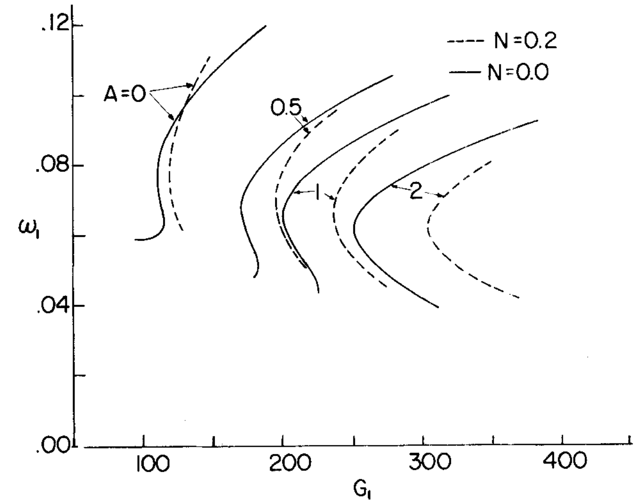

Fig. 11. Downstream disturbance amplification for $Sc = 0.2$, in terms of G_1.

here. The frequencies or $\omega G^{1/3}$ chosen, for example, for $Sc = 2.0$, $N = +0.5$, and $Sc = 0.2$, $N = +0.2$, are close to the most amplified ones downstream in each case. For each Sc and N, the disturbance distributions were calculated both at the neutral condition ($A = 0$) and downstream at the same physical frequency, that is, at $\omega G^{1/3}$ = constant, in the highly unstable region. These ω, G conditions are located in Figures 8 and 10 for the two Sc and N conditions, respectively.

The form of the disturbance amplitude distributions were found to remain the same along constant $\omega G^{1/3}$ paths downstream. The phase speed is also little changed over a wide range of G. This behavior is very similar to that previously found for flow due to a single buoyancy mechanism. This supports the use of the integral, Equation (12), to calculate A, since it is assumed there that the material which possesses the local maximum of disturbance amplitude remains the same downstream.

The changes in the maximum amplitude of s and d are compared to ϕ' as follows:

$$R_1 = \frac{|\phi'|_{\max}}{|s|_{\max}} \quad \text{and} \quad R_2 = \frac{|\phi'|_{\max}}{|d|_{\max}}$$

Our calculations show, for $Sc = 2.0$ and $N = +0.5$, that both R_1 and R_2 increased by about 20% between $G = 136$ ($A = 0$) and $G = 386$ ($A \approx 4$). From the generalization we see that ϕ' must be multiplied by $\nu G^2/4x \propto G^{2/3}$ to find the velocity level at a given x. However, s and d are multiplied by $(t_o - t_\infty)$ and $(C_o - C_\infty)$ to obtain physical magnitude. Thus, the downstream growth of velocity disturbance amplitude is calculated to be 2.4 times as great. The phase changed very little from $A = 0$ to $A \approx 4$, indicating that the actual disturbance propagation velocity remains essentially a constant multiple of the local convection velocity.

We also found that the maxima of the velocity and temperature disturbances occur at about the same value of η for both values of G, or downstream locations. The phase distributions are very similar. The maximum in d moves toward smaller η. This follows from both changes in the phase of d and in its magnitude. This shift is presumably connected with the effect of $Sc < Pr$ in thinning the concentration boundary layer. However, it is not possible to predict this result from the role of d in the stability equations. Perhaps there would also be greater changes in the ϕ' and s distributions for larger N.

The results for $Sc = 0.2$, $N = +0.2$ also indicated even less downstream change in the forms of the disturbance amplitude functions, along $\omega G^{1/3}$ = constant. We may conclude from these small changes that the downstream integration of amplitude, in terms of α_i at given physical frequency, to obtain values of A, is a reasonable approximation in this linear range of disturbance amplification. Experimental measurements have repeatedly supported such results in single buoyancy mode flows.

We also found that the Lewis number determines the relative position, in η, of the maxima of the d and s distributions. For $Le < 1$, the maximum in d is at a larger η than that in s and for $Le > 1$ is at smaller η. For $Le < 1$, the disturbance quantities s and d follow, qualitatively, the distribution of the flow quantities T and C in relative spatial extent. However, the detailed behavior of the disturbance function was strongly dependent on N. The disturbance functions obtained for $Sc = 0.94$ (carbon dioxide diffusing into air) as well as those for $Sc = 2.0$, $N = 0.5$ follow the general trends discussed above.

CONCLUSION

The occurrence of a second buoyancy inducing diffusion process may cause major alterations of the stability characteristics of the resulting flow. Our calculations show very complex interactions between disturbances in velocity, temperature, and concentration. For the range of Lewis number and ratio of the two buoyancy components considered, the additional buoyancy mode alters the disturbance growth characteristics and shifts the frequency range of the most highly amplified disturbances.

The frequency shift is most apparent when the stability characteristics are expressed purely in terms of the thermal Grashof number. This corresponds to a flow circumstance in which the temperature difference $(t_o - t_\infty)$ is held constant while one observes the influence of an added mass diffusion on disturbance filtering and growth. However, if we express the stability plane in terms which more accurately reflect the vigor of the flow, for example, a Grashof number composed of the unweighted sum of the thermal and chemical buoyancy components, these effects progressively disappear as the Lewis number approaches 1. However, the effects remain for $Le = 2.9$ and 0.29, and the flow is less stable when the two buoyancy effects are opposed.

Our results are expected to be realistic estimates of the stability characteristics of actual flows because of the detailed past success of linear stability theory, compared to many experiments, in its predictions of disturbance growth rates, filtered frequency, etc., in flows in an unstratified ambient medium and most recently by Jaluria and Gebhart (1974*b*) in a stratified one. The small downstream variation found here in disturbance form leads us to expect that the integrations of disturbance amplitude are again realistic.

There is the remaining question of how one may accurately estimate the effective local vigor of a flow induced by combined buoyancy modes. The effect of Lewis number on the respective spatial extents of the diffusion layers, and the differing resultant modification of the form of the velocity field, makes this a very complicated and still unanswered question.

The methods used here may be directly applied to other important combined buoyancy mode flows. Both bounded and unbounded flows of great practical importance arise through combined modes and in many different fluids.

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NOTATION

A = disturbance amplification factor, Equation (12)
C = concentration of the diffusing species
C = $(C - C_\infty)/(C_o - C_\infty)$
C' = concentration disturbance
c_p = specific heat
c = disturbance velocity
D = mass transfer diffusivity
d = concentration amplitude function
f = similarity stream function
G = $4\sqrt[4]{Gr'/4}$
Gr' = $PGr_{x,t} + QGr_{x,c}$
$Gr_{x,t}$ = $g\beta x^3 (t_o - t_\infty)/\nu^2$
$Gr_{x,c}$ = $g\beta^* x^3 (C_o - C_\infty)/\nu^2$
g = gravitational acceleration
k = thermal conductivity
Le = α_t/D, Lewis number
$\dot{m}$ = mass diffusion flux
N = ratio of $Gr_{x,c}$ to $Gr_{x,t}$
Nu_x = local Nusselt number
P = constant
Pr = ν/α_t, Prandtl number
Q = constant
Sc = ν/D, Schmidt number
Sh_x = local Sherwood number
s = temperature amplitude function
T = $(t - t_\infty)/(t_o - t_\infty)$
t = temperature
t' = temperature disturbance
u = vertical velocity component
V = characteristic velocity
v = horizontal velocity component

Greek Letters

α = wave number and amplification rate
α_t = $k/\rho c_p$ thermal diffusivity
$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t}\right)_{p,C}$ = volumetric coefficient of thermal expansion
$\beta^* = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial C}\right)_{p,t}$ = volumetric coefficient of expansion with concentration C
δ = boundary region thickness
$\eta = \eta(x, y)$ = similarity space variable
μ = viscosity
ν = μ/ρ kinematic viscosity
ρ = density
Φ = velocity amplitude function
Ψ = stream function
Ψ' = disturbance stream function
ω = disturbance frequency
ω_1 = see Equation (11)

Subscripts

c = concentration
i = index of particular chemical species
o = condition at surface, that is, at $y = 0$
t = temperature
∞ = condition in the remote ambient medium

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APPENDIX

TABLE 1. BASE FLOW QUANTITIES

Sc	N	$f''(0)$	$-T'(0)$	$-C'(0)$	f_∞	L	η_{edge}	$\Delta\eta$
0.94	0.5	0.9067	0.5455	0.6312	0.6486	0.3000	10	0.02
—	0.2	0.7727	0.5194	0.6006	0.6241	0.2619	10	0.02
—	0.0	0.6789	0.4995		0.6059	0.2353	10	0.02
—	−0.2	0.5805	0.4768	0.5506	0.5857	0.2076	10	0.02
—	−0.5	0.4211	0.4348	0.5010	0.5501	0.1634	10	0.02
—	−0.8	0.2371	0.3735	0.4283	0.5047	0.1141	12	0.02
2.0	0.5	0.8736	0.5308	0.8675	0.6238	0.2697	12	0.01
—	−0.5	0.4624	0.4590	0.7332	0.5850	0.1982	12	0.01
0.2	0.2	0.8035	0.5427	0.2728	0.8210	0.3422	20	0.05

For a plane plume flow and for a flow adjacent to an isoproperties surface, Gebhart and Pera (1971) tabulated the integral quantities

$$I_t = \int_{-\infty}^{+\infty} f'T d\eta \qquad I_c = \int_{-\infty}^{+\infty} f'C d\eta$$

However, it is more interesting to note that for the surface flow

$$\int_0^\infty T'' d\eta + 3Pr \int_0^\infty fT'' d\eta = 0$$

from which

$$[T']_0^\infty - 3Pr \int_0^\infty f'T d\eta = 0$$

$$-T'(0) - 3Pr \frac{I_t}{2} = 0$$

or

$$I_t = -\frac{2}{3} \frac{T'(0)}{Pr}$$

Similarly, one finds

$$I_c = -\frac{2}{3} \frac{C'(0)}{Sc}$$

We need not include I_t and I_c in a data tabulation which includes $T'(0)$ and $C'(0)$. Also, integrating (1), the momentum equation for the unperturbed flow, one finds

$$f''(0) + \frac{5}{2} L = \int_0^\infty (T + NC) d\eta$$

where

$$L = \int_{-\infty}^{+\infty} f'^2 d\eta$$

Table 1 lists L and $f''(0)$, so that we know the integral of the buoyancy term.

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On the Fate of Fuel Nitrogen During Coal Char Combustion

The physical and chemical characteristics that influence the conversion of fuel nitrogen to nitrogen oxides during coal char combustion were theoretically examined by using a simplified model in which nitric oxide is an intermediate product between fuel nitrogen and N_2.

It was found that diffusion-reaction interactions were important in determining the selectivity of the char particle toward nitric oxide production. At low temperature fluidized bed combustion conditions, pore size is important, and low conversion of fuel nitrogen to nitric oxide is favored by long narrow pores. Under high temperature, pulverized coal combustion conditions, the model provided insight into mechanisms of nitric oxide formation and predicted the observed weak temperature dependence of fuel nitrogen conversion, as well as a significant effect of particle size.

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